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SOME ASPECTS OF CONSTRUCTIVE MATHEMATICS  
THAT ARE RELEVANT TO THE FOUNDATIONS OF  
NEOCLASSICAL MATHEMATICAL ECONOMICS  
AND THE THEORY OF GAMES

by

✓Alain A. Lewis

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Alain A. Lewis\*\*

1. A Constructive Counterexample To A Non-Constructive Theorem On Demand Correspondences.

Typically, one finds the following mathematical model in the neo-classical literature of mathematical economics e.g. in Debreu [1959], Nikaido [1968] or more recently the article by Muhkerji in Econometrica Vol. 45, No. 4 May 1977.

Definition 1: Let  $E^n$  denote n-dimensional Euclidean Space and  $S \subseteq E^n$ , a non-empty set of all possible alternatives, on which a binary relation  $R$  is defined. Let  $R$  be reflexive, transitive and complete. Then  $R$  is a typical weak preference relation, from which a strict preference relation  $P$  and an indifference relation  $I$  may be obtained in the customary fashion.

Definition 2: Let  $\chi$  denote the class of all non-empty compact subsets of  $S$ . For every  $A \in \chi$ , let

$$C(A) = [x \in A : y \in A \Rightarrow xRy]$$

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If  $C(A) \neq \emptyset$  for every  $A \in \chi$ , then a choice function exists on  $\chi$ .

Definition 3: Let  $\beta$  denote the class of budget sets on  $S$ , where  $\beta \subseteq \chi$ . If a choice function exists on  $B$ , then a demand correspondence exists on  $\beta$ . Let

$$d(B) = \{y \in B : x \in B \Rightarrow y R x\}$$

denote the induced demand correspondence for a fixed element  $B \in \beta$

Theorem 1: If the set  $R_x = \{y : y R x \text{ \& } y \in S\}$  is closed relative to  $S$  for all  $x \in S$ , then  $d(B) \neq \emptyset$  for all  $B \in \beta$  iff for all  $B \in \beta$  and for every finite subset  $\{x_1, \dots, x_n\} \subseteq B$ ,  $\exists y \in B$  s.t.  $y R y$  and  $y R x_j$  for all  $j \leq n$ .

Pf. The only if direction of this is trivial. For sufficiency, let  $R_x(B) = R_x \cap B$ , and for arbitrary  $B$  choose a finite set  $\{x_1, \dots, x_n\} \subseteq B$ . By hypothesis,  $\bigcap_{j \leq n} R_{x_j}(B) \neq \emptyset$ , and the family  $\{R_x(B) : x \in B\}$  has the finite intersection property. Since  $B$  is compact,  $d(B) = \bigcap_{x \in B} R_x(B) \neq \emptyset$ .

This theorem is a typical existential statement in mathematical economics. It tells us that a mathematical object exists having "very nice" properties, but there is no mention of how one might go about finding the set defined by  $d(B)$  for any particular  $B$  in the family of budget sets. In other words, the theorem does not provide us with an algorithm to construct the order-maximal elements from any given set in

B in accordance with the "rules of correspondence" provided by the mathematical object  $d:B \rightarrow B$ .

There are at least two important ways to demonstrate the non-constructive character of the above theorem within recursion theory:

First, the proof of this theorem like many in mathematics uses a highly non-constructive mathematical principle: the Heine-Borel Theorem. In fact, if the Heine-Borel Theorem were restricted to be constructive in a precise interpretation within the framework of recursive analysis, it would not be a theorem at all. We believe that Ernst Specker was among the first to realize this along with his early observation that the Bolzano-Weirstrass Theorem fails in recursive analysis. Thorough discussions of this type of phenomenon can be found in the very comprehensive book on constructive mathematics by Michael Beeson [1985] and in the monograph Structure and Complexity by Lewis [1986].

Secondly, even if the theorem were true constructively, e.g. put in terms of finite sets entirely, the hypothesis of the theorem is not sufficient to provide any effective means for uniformly determining values of  $d(B)$  from effective enumerations of elements in its domain. Here, we call an enumeration effective if it can be carried out by an algorithm; and the set of objects thus enumerated is termed recursively enumerable.

To underscore the second point, here is an example of a non-computable choice function on a recursively enumerable well-defined family of finite sets.

Example 1: We will assume Church's Thesis and identify with any computable function some Turing machine that represents the recursive function and provides the algorithm for the computation of the function. A discussion of Church's Thesis and the terminology and concepts used from recursion theory may be found in Rogers [1967], or Beeson [1985]. We will not go into detail on the reasons why Church's Thesis is a useful and reasonable principle, as we have discussed it elsewhere, cf. Lewis [1985].

Let  $\omega$  be the first infinite ordinal, i.e.

$$\omega = \{0, 1, 2, \dots, n, \dots\}$$

and define the following family of finite sets. For each  $n < \omega$ , let  $B(n)$  be the initial segment of  $\omega$  determined by all integers less than or equal to the maximum number of one's printed by a Turing machine having  $n$ -states. There are infinitely many such machines, each determined by a list of instructions for the computation it performs. But there are only finitely many equivalence classes over the different programmes for the class of  $n$ -state Turing machines. Let the relation  $R$  be determined by the reflexive order on the natural numbers so that  $x R y$  iff  $x \geq y$  for all  $x, y < \omega$ . It is clear that  $|B(n)| < \omega$ , for each

$n$  and so, the set  $\{y: x \in B(n) \Rightarrow yRx\}$  exists for all  $n$  and is simply the quantity  $\operatorname{argmax}_{B(n)} f$  for  $f$  the identity function on  $\omega$ . Note that the identity function is a recursive function; in fact, it is a primitive recursive function. And so, for each  $n$ , the quantity  $\operatorname{argmax}_{B(n)} f$  may be effectively found.

But now let us ask the question whether there exists a recursive function  $h \in {}^\omega\omega$  such that for all  $n < \omega$ ,

$$h(n) = \operatorname{argmax}_{B(n)} f$$

First of all, if there were such a function  $h \in {}^\omega\omega$ , it is easy to see that there are certain features it must possess.

Lemma 1: If there exists a recursive function  $h \in {}^\omega\omega$  such that for all  $n < \omega$ ,  $h(n) = \operatorname{argmax}_{B(n)} f$  then,

$$(1) \quad h(1) = 1$$

$$(2) \quad h(n+1) > h(n)$$

$$(3) \quad h(n+1) \geq 2n$$

Pf: Boolos and Jeffrey [1974] or T. Rado [1962]

Theorem 2: There is no recursive function  $h$  satisfying properties (1) - (3).

Pf: It is not hard to see that if  $h$  were recursive then, for  $k$  the # of states in the Turing machine that computes  $h$

$$h(n+2k) \geq h(h(n))$$

(cf. Boolos and Jeffrey [1974] p. 38). Now the property that  $h(n+1) > h(n)$  yields that  $h(j) > h(i)$  if  $j > i$  for all  $i, j < \omega$ . Thus,  $j \leq i$  if  $h(j) \leq h(i)$  for all  $i, j < \omega$ . Now let  $i = n + 2k$  and let  $j = h(n)$  to obtain

$$n + 2k \geq h(n) \text{ for all } n < \omega$$

Positive translations do not affect this relation, and so,

$$n + 11 + 2k \geq h(n + 11) \text{ for all } n < \omega$$

follows if  $h$  is recursive.

From the lemma, part (3)  $h(n + 11) \geq 2n$  (and this is true whether or not  $h$  is recursive). Combining inequalities, we obtain

$$n + 11 + 2k \geq 2n$$

if  $h$  is recursive. Therefore,

$$11 + 2k \geq n$$

for all  $n < \omega$ , if  $h$  is recursive. In the particular case of  $n = 12 + 2k$ , this relation gives the following piece of nonsense:

$$11 + 2k \geq 12 + 2k$$

Therefore the function  $h$  cannot be recursive

It is not hard to construct other examples of non-computable choice functions for any recursively enumerable strictly monotone

increasing family of finite sets. We have chosen the above example for the sake of its clarity.

The fact that no recursive  $h$  can exist such that

$$h(n) = \operatorname{argmax}_{B(n)} f$$

for all  $n$ , means that the demand correspondence induced by the choices made in accordance with  $f$  cannot be recursively realized in a uniform way on the family of finite sets  $\{B(n) : n < \omega\}$ . The paper entitled "On Turing Degrees of Walrasian Models", Lewis [1987] gives a formalization of this example in terms of the partial ordering on the Turing degrees of unsolvability as an extension of our earlier work on recursively representable choice functions. The graph of any such function that uniformly computes

$$\operatorname{argmax}_{B(n)} f$$

for all choices of  $n < \omega$  cannot even be R.E. As we have argued elsewhere, this level of complexity is exceedingly high for the realization of recursively representable choice functions, as it places the matter of effectively realizing recursively representable choice functions in excess of Hilbert's Tenth Problem for the decision procedure of Diophantine predicates over the integers (c.f. Matiyasevic [1970]). What this means is that even if we had a machine or an algorithm that could provide solutions to Hilbert's Tenth Problem, said

machine or algorithm would not come close to the uniform realization of a recursively representable choice function in neoclassical mathematical economics.

The above counterexample, and the resulting complexity of recursively representable choice functions, taking place in the domain of countable families of finite sets, leads us to ask the following questions: Do non-trivial demand correspondences really exist in any meaningful (i.e effectively) constructive sense? Within the confines of Church's Thesis and its attending equivalences to the recursive functions, the answer seems to be, no.

## 2. Implications Of Non-Recursive Realizability For The Complexity Of Walrasian Models.

In the paper, "On Turing Degrees of Walrasian Models", we represent a model of Walrasian general equilibrium as a two sorted structure:

$$\alpha = \langle R^{(n+m)\ell}, I, J, \{(X_i, \lambda_i)\}_{i \in I}, \{(Y_j, \eta_j)\}_{j \in J} \rangle$$

$\ell$  is the dimension of the commodity space and the structure has two sorts of variables:  $I$  of cardinality  $m$  for consumer agents, and  $J$  of cardinality  $n$  for producing agents, along with sets of criterion functions for each type of agent,  $\{(X_i, \lambda_i(w_i, p))\}_{i \in I}$  and  $\{(Y_j, \eta_j(p))\}_{j \in J}$  as defined in Debreu [1959].

Intuitively speaking, a choice function on a recursive domain is recursively realizable if and only if its graph is recursively

solvable, i.e. its graph is a recursive set in the appropriate product space of a choice of recursive metric space. Since the complexity of the realization of any recursively presented model of Walrasian general equilibrium in the sense of Turing equivalences can be no less than the complexity of the realization of the choice functions for each type of agent, the non-recursive realizability of recursively representable choice functions on finitely dimensioned Euclidean domains implies the non-recursive realization of models of Walrasian general equilibrium that are recursively presented, with trivial models being the only exception.

These matters may be summarized by the following results.

Theorem 3: Let  $\langle R(X), F_R \rangle$  be a recursive space of alternatives derived from the recursive metric space of  $R^n$ ,  $M(R^n)$  for  $R(X)$  the recursive representation of a compact, convex subset of  $R^n$ . Let  $C: F_R \rightarrow F_R$  be a non-trivial recursive rational choice on  $\langle R(X), F_R \rangle$  and select from the class of sequences  $(F_R)^\omega$  any non-null element  $\{^F R_j\}_{j < \omega}$  with infinitely many distinct terms for the domain of  $C$ . Then per fixed selection of  $\{^F R_j\}_{j < \omega}$ , the co-domain of graph  $\langle C \rangle$  is non-recursive and therefore the choice function is not recursively solvable and thus cannot be recursively realized.

Corollary: No non-trivial recursively representable model of Walrasian general equilibrium or N-person non-cooperative game in the sense of Nash is recursively realizable.

3. Are There Totally Effective Frameworks For Mathematical Economics?

Recursion theory is not synonymous with the contemporary theory of computational complexity and so one may ask what the implications of a Turing degree classification result within the Kleene-Mostowski arithmetic hierarchy are for polynomially bounded computations. We briefly discuss these implications in this final section.

First of all the Polynomial hierarchy in computer science is an adaptation of the Kleene-Mostowski hierarchy by restricting quantified formulae to be polynomially bounded statements. It is not known however if the Poly-hierarchy is distinct or whether it collapses at some level of complexity, as a consequence of  $P = NP$  or the fact that complete NP sets are polynomially isomorphic. Still, one may inquire to what extent is it possible to "transfer" results in Kleene-Mostowski hierarchy "downwards" to the Poly-hierarchy.

R. Jerislow [1985] has obtained results of complexity for very simple-minded equilibrium models that are based upon Stackleberg [1934] sequenced-move games placed in a setting of multi-level integer programmes.

Theorem 4: (Jerislow [1985] ) The class of multi-level finite Stackleberg models of equilibrium is PSPACE complete

Thus, even in the case of simple-minded translations of models of economic equilibrium into the framework of the Poly-hierarchy the bounds of complexity seem in excess of any NP-complete problem, e.g. integer programming, the existence of Hamiltonian circuits in a finite

graph, the sub-graph isomorphism problem for finite graphs, finite graph  $k$ -colorability, existence of equilibrium points for finite  $N$ -person non-cooperative games with product-polynomial payoffs, and a host of other problems that can be found in Garey & Johnson [1979]. Problems in the theory of algorithms that are also PSPACE complete are (i) the proper representation of regular expressions over the binary alphabet  $\{0,1\}$ ; (ii) The Meyer-Stockmeyer [1973] problem of determining whether two person alternating games on quantified Boolean formulas are determined. (iii) the existence of winning strategies in E.H. Moore's game of  $k$ -Nim or (iv) the existence of winning strategies in the simple game of two-dimensional Hex:



Figure 1

On the other hand, if we try to translate the mathematical framework of the more complex Walrasian models into the Poly-hierarchy, then many difficulties that are conceptual in character arise. The first such difficulty is that the Poly-recursive reals do not form a

recursive field. This is a consequence of a result due to Jockusch of the University of Illinois at Urbana to the effect that the primitive recursive reals do not form a recursive field, i.e. neither of the above sets of numbers with the appropriate operations and distinguished elements can be isomorphic to any recursively presented field. Another way of highlighting the defect of considering only Poly-recursive reals is that Moschovakis [1964] has shown that the recursive metric space we have employed to obtain recursive presentations of Walrasian models of general equilibrium are recursively categorical for the class of listably-ordered recursively presented fields, i.e. the recursive metric space  $M(\mathbb{R}^n)$  is recursively isomorphic to any other countable listably-ordered, recursively separable, recursive field.

Other barriers to doing classical analysis on just the Poly-recursive reals are detailed in the recent work of Harvey Friedman [1985] where it is shown that simple closure properties of the Poly-recursive reals with respect to maximization entail the collapse of the Poly-hierarchy through the consequence that  $P = NP$ . Thus any downward transformation of the Kleene-Mostowski hierarchy to the polynomially-bounded predicates vanishes if we require algebraic closure for the Poly-recursive real numbers.

As things stand, we do not know how to formulate the necessary analogues within the Poly-recursive real numbers to obtain Poly-recursive representations of Walrasian models of general equilibrium. And even if this were accomplished in some acceptable way, for the

purpose of ranking realizations of Poly-recursive representable structures, one needs some classification of the relevant sets of Poly-recursive reals in terms of the Poly-m degrees, i.e. those degrees generated by the reducibility:  $\leq_m^P$ , where the  $P$  and  $m$  represent Polynomially-many-one reducible. But these degrees are much more complex than the Turing degrees and one does not have simple relationships between orderings on the Poly-m degrees and levels of complexity within the Poly-hierarchy, and for the reducibility to make sense, not only do we have that a set of integers  $A$  in  $\omega$  is  $\leq_m^P$  - reducible to another set  $B \subseteq \omega$  if for some polynomially bounded recursive function  $f$ ,  $x \in A$  iff  $f(x) \in B$  but we must require that  $f$  have a polynomial inverse as well. These special features of the Poly-degrees lead to the fact that the theories of the R.E. sets and the NP sets are not elementary equivalent from differences in the resulting structures induced by the special features of the Poly-degrees. A full discussion of this can be found in the paper by P. Odifreddi, "Recursion-Theoretical Aspects of Complexity Theory", Department of Mathematics, Cornell University, 1985.

Our results within the Turing degree framework of the Kleene-Mostowski hierarchy and the above mentioned difficulties in finding suitable translations of recursive representability within the Poly-hierarchy leads to another issue.

One way to interpret our results is to say that the notion of a recursively presented field is too complex an algebraic object to serve

as a framework for an effective theory of games. On one hand if one gets recursive presentations of fields, the relevant task-correspondences cannot be recursively realized, and if one tries to put things within realm of feasible computations by restricting the alternatives and outcome spaces to just sets of Poly-recursive reals, we do not get enough algebraic closure to carry out the theory.

Faced with this situation, it seems reasonable to search for algebraic objects that are less complex than recursively presented fields. For example, if we restrict games to be played on arithmetically definable sets of integers, life is more pleasant from the standpoint of computational complexity. These models translate easily into subrecursive fragments of arithmetic, and reasonable bounds for the Poly-recursive games that are played take place in PSPACE. The relevant algebraic structure here is a discretely ordered ring. Actually, in most game-theoretic models, the full-force of the field of real numbers is not required, and the choice of the real numbers for Walrasian models of general equilibrium comes from the desirable topological properties  $\mathbb{R}$  enjoys as an ordered structure.

Of course, if one simplifies the admissible algebraic structures that games are to be played upon, it becomes more difficult to obtain what we like to call "good" results. Typically, the simpler a structure one deals with mathematically, the more complex the techniques employed to obtain deep results. The paradigm of this is of course number theory, or the theory of finite groups vs. the theory of functions of a

real variable, or functional analysis. Evidence to support this view of ours is the fact discussed above that one cannot "transfer" downwards to the Poly-hierarchy, in a uniform way, the power of the topological techniques of analysis. Of course, finite fields do exist e.g. the Galois fields, but here a rather careful understanding of algebraic curves, transformation groups and collineations is required to look into the possible use of such fields within the projective geometries that are defined over them. I have not assessed the sophistication of the typical graduate course in mathematical economics lately, but I suspect this programme, and any such like it, would have to extend well into the next decade for any widely based acceptance by the profession.

If, for the sake of computational viability, the approach of simplifying algebraic structures is taken as a methodology for modelling economic theory, one need not go all the way to the extreme case of considering totally finite models, with no infinity present in the alternative space or outcome space, as Campbell [1976] has done, to obtain recursively realizable choice functions or Walrasian models. For example, suppose it were possible to predicate a theory of games (and Walrasian models of equilibrium) on the consequences of a positive solution to Hilbert's 10th problem. A very important mathematical result of last year by Robert Rumely of the University of Georgia tells us that such a theory could be effectively carried out over the ring of all algebraic integers of a finite extension field of the rationals. If one took the ring of all algebraic integers as a basis for game-

theoretic structures and allows the space of alternatives and the space of outcomes to be recursive sequences of codes into the ring of algebraic integers, and further allows only operations that are definably equivalent to the solutions of Diophantine equations in the task correspondences of the structures, then Rumely has shown that based on an earlier work of Cantor & Roquette [1984], fully effective procedures exist to realize such correspondences. What Rumely has shown is that there is a primitive recursive decision procedure to realize Diophantine predicates over recursive subsets of the ring of all algebraic integers. In addition to this tremendously important result, recent work by Manders & Adelman [1980] and [1981] have shown that polynomially recursive decision procedures are available for restricted forms of Diophantine predicates over  $\omega$ .

So, our personal preference to obtain a fully effective theory of games is the approach of restricting effective constructions in the theory of games over decidable predicates of simpler algebraic structures than the recursively presented fields. The way in which one would carry out the development of a theory in such a setting would be simply to try and prove as many theorems as possible about the structures that are combinatorial in nature and that follow from the realizations of the task-correspondences that are definable in terms of the decidable predicates of the structure. A very good beginning can be had with the study of Diophantine predicates over the ring of algebraic integers. We have recently shown that Arrow's impossibility Theorem can

be extended to the infinitary setting in an effective way by using the recursive complexity of Diophantine predicates over the ring of algebraic integers within the lattice of its R.E. sets. Using von Neumann simple games (cf. Shapley [1962]) without the use of the algebraic integers as a coding device, we have shown that Arrow's Impossibility Theorem (cf. K.J. Arrow [1951]) can be extended to the infinitary setting using the R.E. complexity of Diophantine predicates over the positive integers only (i.e. the natural numbers). For details see the forthcoming research announcement: "An Infinite Version of Arrow's Theorem In The Effective Setting", Lewis [1986], to appear in Mathematical Social Sciences. More recently, using similar techniques, we have obtained a very interesting positive recursion-theoretic result for the abstract allocative mechanisms initiated by Leonid Hurwicz [1960]: among the class of recursively presentable Hurwiczian discrete allocative mechanisms,  $\sigma = \{\sigma_j\}_{j < \omega}$ , there is an R.E. class,  $\tilde{\sigma} = \{\tilde{\sigma}_j\}_{j < \omega}$ , whose performance correspondences are uniformly  $\Delta^{\circ}_2$ -decidable; and within  $\tilde{\sigma}$ , there is an R.E. subclass  $\mathcal{L} = \{\mathcal{L}_j\}_{j < \omega}$  whose performance functions are uniformly recursively realizable.

Actually, the theorem we prove is actually stronger. In effect, we show that there exists an R.E. class of resource allocation mechanisms which is uniformly realizable in NP-complete complexity. The reader is asked to note that the class constructed is a class of resource allocation mechanisms whose realizations use uniformly sub-recursive! Of course by our previous results, these mechanisms cannot be

Walrasian, but we have made the observation that the following type of Hurwiczian mechanism does in fact satisfy the description provided in our theorem.

Let  $\pi$  be a Hurwiczian mechanism of the form:

$$\pi = \langle E, M, A, (h, g) \rangle$$

where  $E$  is an environment,  $M$  a message space  $h: ExM \rightarrow A$  is an outcome function for  $A$  a space of actions and  $g: ExM \rightarrow Z$  is a performance criterion. We require in this setting that  $E, M$  and  $A$  are discrete, i.e. we allow  $E \cong M \cong A \cong Z$  where  $Z = \{\dots, -n, -(n-1), \dots, -2, -1, 0, 1, 2, \dots, n, n+1, \dots\}$  Obviously the complexity of a mechanism such as  $\pi$  resides in the choice of  $h$  and  $g$ . Of course  $E = \prod_{i \in I} E_i$  and  $M = \prod_{i \in I} M_i$  for  $I = \{1, \dots, n\}$  a set of agents and  $g = (g_1, \dots, g_n)$  while  $h = (h_1, \dots, h_n)$ .

Now suppose equilibrium outcomes for the mechanism  $\pi$  are given by a set of equations

$$\{g_i(e_i, m_i) = 0\}_{i \in I}$$

where by obvious convention  $g_i: E_i \times M_i \rightarrow Z$ . Hurwicz construes the performance of a structure such as  $\pi$  by means of a performance correspondence  $F: E \rightarrow A$  given by the rule:

$$F(e) = \{a: \exists \langle e, m \rangle \in ExM [\forall i \in I [g_i(e_i, m_i) = 0 \ \& \ h_i(e_i, m_i) = a_i]]\}$$

It simplifies matters greatly if we allow  $|A| = |A_1| \times \dots \times |A_n| = K < \omega$ , but the following result is true if  $A_i = Z$  for all  $i \in I$ . This Theorem, we believe, is the first result of its kind that establishes a clear and solid concrete link between the sub-recursive complexities of contemporary computer science, and an entire class of resource allocation mechanisms that are comparable in task to Walrasian models.

Theorem 5: For the class of discrete Hurwiczian allocation mechanisms  $\alpha = \{\langle E_\alpha, M_\alpha, A_\alpha, (h, g)_\alpha \rangle_{\alpha \in \tilde{A}}\}$  for  $\tilde{A}$  some infinite index set, there exists an R.E. subclass of recursively presented mechanisms  $\tilde{\alpha} = \{\langle E_j, M_j, A_j, (h, g)_j \rangle = \pi_j\}_{j < \omega}$  such that the associated performance criteria for the class  $\tilde{\alpha}, \{F_j(e)\}_{j < \omega}$ , is a uniform recursive class of functions, which is uniformly realizable in NP-complete complexity.

For the associated performance criteria class  $\{F_j(e)\}_{j < \omega}$  of the structures  $\tilde{\alpha}$ , the task of finding actions  $a \in A$  for a given  $e \in E$  is an NP-complete problem; in other words, if we know the computation of any NP-complete problem in computer science, then this computation is sufficient to obtain  $F_j(e) = a$  by a computation for all  $j < \omega$ ! We feel that the discovery of the class  $\tilde{\alpha}$  will allow a general theory of resource allocation that is realistically computational in character over a very rich class of Hurwiczian mechanisms

$\tilde{\alpha} = \{\langle E_j, M_j, A_j, (h, g)_j \rangle = \pi_j\}_{j < \omega}$  that have recursive presentations.

What this means in turn is that properties of the models  $\alpha_j \in \tilde{\alpha}$  that are predicated on whether  $F_j(e) = a$  or  $F_j(e) \neq a$  can be checked by the solutions to an NP-complete problem.

Here are three concrete examples of NP-complete problems that will each provide enough mathematical information to realize the class of performance correspondences  $\{F_j(e)\}_{j < \omega}$  for structures  $\pi_j \in \tilde{\alpha}$  in an effectively computable way.

Problem 1: Let  $\{G_j : 1 \leq j \leq n\}$  be a set of product polynomials over  $Z[x_1, \dots, x_n]$  such that  $\text{Rng}(G_j) \subseteq M_j \subseteq Z$  and  $M_j$  is a finite set for all  $1 \leq j \leq n$ . Does there exist a sequence  $\langle y_1, y_2, \dots, y_n \rangle$  of integers with  $y_j \in M_j$  for all  $1 \leq j \leq n$  such that for all  $1 \leq j \leq n$  and  $y \in M_j$

$$G_j(y_1, \dots, y_{j-1}, y_j, y_{j+1}, \dots, y_n) \geq G_j(y_1, \dots, y_{j-1}, y, y_{j+1}, \dots, y_n)?$$

Solutions  $\langle y_1, \dots, y_n \rangle$  can be found in NP-complete complexity.

Problem 2: Let  $\langle a_1, \dots, a_n \rangle$  be an arbitrary sequence of integers. Is  $\langle a_1 \theta, a_2 \theta, \dots, a_n \theta \rangle$  a solution to

$$\int_0^{2\pi} \sum_{j=1}^n (\pi \cos(aj\theta)) d\theta = 0 ?$$

Problem 3: Define an integer expression class over the structure  $\mathcal{I} = \langle Z^+, \cup, + \rangle$  inductively as follows: (i) If  $n \in Z^+$ , then the binary expression for  $n = \sum_{j < \omega} 2^{-j} x_j$ , for  $x_j \in \{0, 1\}$  is an integer expression representing  $n$ . (ii) If  $f$  and  $g$  are integer expressions that represent sets  $F$  and  $G$  in  $Z^+$ , then  $f \cup g$  represents  $F \cup G$  and  $f + g$  represents  $F + G = \{m + n : m \in F \text{ and } n \in G\}$ .

$n \in G$ ). Let  $K \in Z^+$  and let  $e$  be an integer expression for an arbitrary set  $E \subseteq Z^+$ . Is  $K \in E$ ? Whether or not  $K \in E$  or  $K \notin E$  is answerable with an oracle for an NP-complete problem.

The solutions to any one of the above problems in the form of algorithms will provide enough information (mathematically) to uniformly realize the task of the family of performance correspondences  $\{F_j(e)\}_{j < \omega}$  asserted to exist by Theorem 1.

To construct the class  $\tilde{\sigma}$  we allow the functions  $\{(h, g)_j\}_{j < \omega}$  for a mechanism

$$\pi_j = \langle E_j, M_j, A_j, (h, g)_j \rangle$$

to be of a specific binary quadratic form, i.e., every  $g_i$  in  $g = (g_1, \dots, g_n)$  or  $h_i$  in  $h = (h_1, \dots, h_n)$  has the form  $\alpha x^2 + \beta y = c$  for choice of parameters  $\{\alpha, \beta, c\} \subseteq Z$  where  $c = 0$  uniformly if we are in the subclass  $\{g_i\}_{i \in I}$  and  $c = a_i$  uniformly if we are in the subclass  $\{h_i\}_{i \in I}$ . The definition of

$$F(e) = \{a: \exists \langle e, m \rangle \in \text{ExM}[\forall i \in I [g_i(e_i, m_i) = 0 \ \& \ h_i(e_i, m_i) = a_i]]\}$$

Now reduces to a convenient algebraic expression; it suffices to find the points of the algebraic variety in  $Z^2$  defined by the expression

$$\bigwedge_{i \in I} [(g_i(e_i, m_i)) (h_i(e_i, m_i)) = 0] = \varphi(\pi_j)$$

Here we obtain integral points  $\langle e, m \rangle \in \varphi(\pi_j)$  from a product of zeros from binary quadratic forms  $\alpha x^2 + \beta y = c$  over  $Z^2$ . Finding zeros of

the binary forms  $\alpha x^2 + \beta y = c$  is an NP-complete problem in each instance (the proof of this is rather deep and requires a clever number-theoretic reduction) and so the question of  $\langle e, m \rangle \in \varphi(\pi_j)$ ? is NP-complete for each mechanism  $\pi_j \in \tilde{\alpha}$ . Just code the programmes for each equation into a "system programme"  $\varphi(\pi_j)$  by dovetailing. It is worth mentioning that the binary quadratics above are not the only classes of number-theoretic functions that will give similar results in complexity. The result is still valid if the functions  $\langle g_j, h_j \rangle$  have polynomial form in  $Z[x_1, \dots, x_n]$  for arbitrary  $n < \omega$  if only one dimension is coded in a non-linear way.

The importance of Theorem 5 is that it enables comparisons of complexity with other models of games in the sub-recursive hierarchy, i.e. the polynomial-hierarchy. For example, the above R.E. class  $\tilde{\alpha}$  of Hurwiczian mechanisms is uniformly less complex than the following class of game-theoretic models.

$$\mathcal{L} = \langle \{A_j^i\}_{i \in I}, \{x_j^i\}_{i \in I}, h_j, \{c_j^i\}_{i \in I} \rangle_{j < \omega}$$

Here the  $\{A_j^i\}_{i \in I}$  are  $m \times n_i$  matrices over  $Q$  and  $x_j^i$  is an  $n_i \times 1$  matrix over  $Q$  and each  $c_j^i \in Q$  is such that  $c_j^i x = \sum_{k=1}^p c_j^{ik} x_j^k$  for  $I = \{1, 2, \dots, p\}$ ,  $p$  a positive integer. The game is played by allowing the set of players to each choose a variable  $x_j^i$  for a given structure

$$\sigma = \langle \{A^i\}_{i \in I}, \{x^i\}_{i \in I}, b_\sigma, \{c^i\}_{i \in I} \rangle$$

with player  $p$  choosing  $x^p$ ,  $p-1$  choosing  $x^{p-1}, \dots$  player 1 choosing  $x^1$ .

Given constraints of the form:

$$\sum_{i=1}^p A^i x^i \leq b_\sigma$$

the game is played in order to maximize  $c^p x$  over values  $x \in S_p$  where  $S_i$ ,  $1 \leq i \leq p$  is defined inductively as the set  $x$  which maximizes  $c^i x$  over  $S_{i-1}$ . The set which satisfies the constraints  $\sum_{i=1}^p A^i x^i \leq b_\sigma$  is denoted as  $S_0$ . Obviously, we assume  $S_0 \neq \emptyset$ . If  $p$  is an arbitrary positive integer, the problem of determining the value for these games is PSPACE-complete! (cf. Jerislow [1984]). This complexity is as high as one can possibly go within the polynomial-hierarchy. And this discussion leads to the next result.

It is known that  $P \neq NP$  implies that PSPACE complexity lies properly above every fixed complexity category of the polynomial-time hierarchy, i.e.  $\sum_n^{0,p} \cup \pi_n^{0,p} \subsetneq PSPACE$ . If  $\alpha$  is a class of structures with recursive presentations, we let  $\maxdeg(\alpha)$  and  $\mindeg(\alpha)$  ( $\bar{deg}(\alpha)$  and  $\underline{deg}(\alpha)$ ) stand for the maximal and minimal degrees of structures  $\sigma \in \alpha$  with respect to the Turing degree of complexity of the realization of the task associated with  $\sigma, \Phi_\sigma$ . For a more detailed view of this kind of model-theoretic analysis, the reader should consult the paper by the author: "On Degrees of Game-Theoretic Structures", Cornell Department of Mathematics, [1986], to appear in

Mathematical Social Sciences [1988] or the monograph, Structure and Complexity, [1986] also by the author.

Theorem 6: Assume  $P \neq NP$ . Then for the class  $\mathcal{L} = \{ \langle \{A_j^i\}_{i \in I}, \{x_j^i\}_{i \in I}, b_j, \{c_j^i\}_{i \in I} \rangle \}_{j < \omega}$  of p-stage multilinear programmes and the R.E. class  $\tilde{\alpha} = \{ \pi_j = \langle E_j, M_j, A_j, (h, g)_j \rangle \}_{j < \omega}$  of Hurwiczian recursively presented allocation mechanisms constructed in Theorem 5,

$$\deg_{+p}(\tilde{\alpha}) < \deg(\mathcal{L})$$

for  $<_{+p}$  a polynomial Turing reducibility.

In terms of algorithms that attain PSPACE-complete complexity or NP-complete complexity results like Theorem 6 allow concrete trade-offs between the performance characteristics of mechanisms and game that use structures in the classes  $\tilde{\alpha}$  and  $\mathcal{L}$  and the degree of computational complexity associated with the realization of those performances. Since most of the problems of PSPACE or NP complexity have very interesting mathematical formulation that are concrete in character, this kind of analysis places the assessment of the complexity of resource allocation mechanisms within very interesting areas of ordinary mathematics, i.e. algebra and combinatorial graph theory. In the case of the recursive structures in  $\tilde{\alpha}$  or  $\mathcal{L}$ , the comparison is quite strong, since the difference in the complexity bound of structures in  $\mathcal{L}$  above those in  $\tilde{\alpha}$  is uniform.

Obviously, similar results should be obtainable for other games specific to mathematical economics, but at a cost of the desirable

consequences of properties inherent in finitely dimensional Euclidean domains, such as convexity and other topological features, which seem to us to be the source of noneffectiveness in the models from the recursion-theoretic point of view. To assess the feasibility of such a programme for mathematical economics, one would have to establish just how much of the "economic" theory can be carried out in totally discrete mathematical setting, as Hurwicz and Marshak [1985] have assayed. Ostensively, it seems to us that any portion of the theory that is essentially dependent upon topological features of finite dimensional Euclidean spaces, or convexity, may be the price of admission to the totally effective setting. For example, in Garey and Johnson [1979] it can be found that the complexity of the existence of equilibrium strategies for Nash N-person noncooperative games played on finite sets of pure strategies and with polynomial payoff functions is PSPACE - complete. This complexity, while in excess of P or NP complexity is well within acceptable bounds for recursive realizability, however.

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